Topology and its Applications 30 (1988) 127-148
North-Holland

ON COMPLETENESS IN QUASI-METRIC SPACES

Doitchin DOITCHINOV
Department of Mathematics, University of Sofia, 1090 Sofia, Bulgaria

Received 5 September 1986
Revised 24 August 1987

A notion of Cauchy sequence in quasi-metric spaces is introduced and used to define a standard completion for a special class of spaces.

AMS (MOS) Subj. Class.: Primary 54E99, 54E52; secondary 54C20

quasi-metric space, completeness, Cauchy sequence, completion

Introduction

A *quasi-metric space* is a set X equipped with a quasi-metric d, i.e. with a nonnegative function d defined on X, X and satisfying the following two conditions:

(a) $d(x, y)=0$ if and only if $x=y$;

(b) $d(x, z)\leqslant d(x, y)+d(y, z)$ for $x, y, z \in X$.

When, in addition, the condition of symmetry $d(x, y)=d(y, x)$ is fulfilled, then d is a *metric*.

A metric space is a special kind of quasi-metric space. On the other hand quasi-metric spaces represent a particular case of quasi-uniform spaces.

A quasi-metric space X with quasi-metric d will be denoted by (X, d). Sometimes, when no ambiguity is possible, the space (X, d) may be denoted simply by X. In particular, when $Y\subset X$, the subspace $(Y, d|_Y)$ of the space (X, d) is usually denoted by Y.

Every quasi-metric space (X, d) can be considered as a topological space on which the topology is introduced by taking, for any $x\in X$, the collection $\{B_r(x) \mid r>0\}$ as a base of the neighbourhood filter of the point x. Here the ball $B_r(x)$ is defined by the equality

$$B_r(x)=\{y\in X \mid d(x, y)<r\}.$$

According to this convention, a sequence $\{x_n\}$ in X is *covergent* to a point $x\in X$, denoted by $x_n\to x$, when $\lim d(x, x_n)=0$.

0166-8641/88/$3.50

It is well known that any topological space is quasi-uniformizable and any T_1-quasi-uniformity with countable base is quasi-metrizable.

A natural problem that arises is to introduce a convenient notion of completeness and to propose a construction of completion of quasi-metric spaces. This construction should be the usual metric completion when (X, d) is a metric space. The analogous problem arises, of course, in a more general setting in the theory of quasi-uniform spaces.

The notion of completeness has to be founded, certainly, on a concept of *Cauchy sequence* which generalizes the concept of convergent sequence. Then a space is *complete* if every Cauchy sequence in it is convergent. A *completion* of a quasi-metric space (X, d) is a complete quasi-metric space (X^*, d^*) in which (X, d) can be quasi-isometrically embedded as a dense subspace.

In problems about completeness and completions, those quasi-metric spaces with a Hausdorff topology are the most important. The assumption that the topology in (X, d) be a Hausdorff topology is equivalent to the following condition concerning the quasi-metric d:

$$\text{if } \lim d(x, x_n) = 0 \quad \text{and} \quad \lim d(y, x_n) = 0, \quad \text{then } x = y.$$

(I.e. it is impossible to have $x_n \to x$ and $x_n \to y$ for $x \neq y$.) The quasi-metrics satisfying this condition will be called *T_2-quasi-metrics.*

Thus the problem is posed as follows. A notion of Cauchy sequence in any T_2-quasi-metric space (X, d) has to be defined in such a manner that the following requirements are fulfilled:

(i) every convergent sequence is a Cauchy sequence;

(ii) in the metric case (i.e. when (X, d) is a metric space) the Cauchy sequences are the usual ones.

Further a standard construction of a T_2-completion (X^*, d^*) of any T_2-quasi-metric space (X, d) should be possible such that:

(iii) if $(X, d_X) \subset (Y, d_Y)$, then $(X^*, d_X^*) \subset (Y^*, d_Y^*)$, where the inclusions are understood as quasi-metric embeddings and the second one is an extension of the former;

(iv) in the case when (X, d) is a metric space (X^*, d^*) is nothing but the usual metric completion of (X, d).

The following simple example shows that this program cannot be fulfilled for arbitrary T_2-quasi-metric spaces.

Example 1. Let $X = (0, 1]$, $Y = [0, 1]$ and $d_X(x, y) = |x - y|$ for $x, y \in X$,

$$d_Y(x, y) = \begin{cases} |x - y| & \text{if } x, y \in Y \text{ and } y \neq 0 \text{ or } x = y = 0, \\ 1 & \text{if } y = 0 \text{ and } 0 < x \leq 1. \end{cases}$$

Then (X, d_X) is a metric space and (Y, d_Y) is a T_2-quasi-metric one. Suppose (X^*, d_X^*) and (Y^*, d_Y^*) are T_2-quasi-metric completions of (X, d_X) and (Y, d_Y), respectively. If the requirement (iv) is fulfilled, then

$$X^*=[0,1] \quad \text{and} \quad d_X^*(x,y)=|x-y| \quad \text{for } x,y\in X^*.$$

But the spaces (X, d_X) and $(Y\backslash\{0\}, d_Y)$ are obviously isometric and thus $(X, d_X)\subset(Y, d_Y)$. (Note, that the restriction of d_Y on $Y\backslash\{0\}$ is a metric.) On the other hand we have $1/n\to 0$ in (Y, d_Y) and $1/n\to 0$ in (X^*, d_X^*). If the requirement (iii) is also satisfied, then from $(X^*, d_X^*)\subset(Y^*, d_Y^*)$ and $(Y, d_Y)\subset(Y^*, d_Y^*)$ (all inclusions understood as quasi-metric embeddings) it follows (since (Y^*, d_Y^*) is a T_2-space) that (X^*, d_X^*) and (Y, d_Y) are quasi-isometric, i.e. that d_X^* and d_Y coincide—a contradiction.

So the requirements (iii) and (iv) cannot be satisfied simultaneously by T_2-completions in the general case of T_2-quasi-metric spaces.

Thus it would be desirable to have a notion of Cauchy sequence which satisfies requirements (i) and (ii) in the general case and that some supplementary conditions may be imposed on the quasi-metric in order to ensure that a completion satisfying (iii) and (iv) can be constructed.

For a metric space the usual definition of Cauchy sequence can be formulated as follows:

The sequence $\{x_n\}$ is called Cauchy sequence if for any natural number k there exists an N_k such that $d(x_m, x_n)<1/k$ for $m, n>N_k$. (1)

It is easy to see that this definition is not suitable for quasi-metric spaces because it does not satisfy in the general case the most basic requirement (i).

With the purpose of expressing the property that a filter in a quasi-uniform space contains "arbitrarily small" sets, the following definition is given in [4] (adopted also in [2] and [1]):

A filter $\mathcal{F}$ in a quasi-uniform space $(X, \mathcal{U})$ is called Cauchy filter if for each $U\in\mathcal{U}$ there is a point $x=x_U$ in X such that $U(x)\in\mathcal{F}$. (Here, as usual, $U(x)=\{y\in X\,|\,(x,y)\in U\}$.)

In the case of a quasi-metric space (X, d) this definition is equivalent to the following one:

A sequence $\{x_n\}$ is called Cauchy sequence if for every natural number k there are a $y_k\in X$ and an N_k such that $d(y_k, x_n)<1/k$ when $n>N_k$. (2)

(In [3] a number of different definitions of Cauchy sequence in quasi-metric spaces are proposed. However only one of them, namely that which coincides with definition (2), satisfies requirement (i).)

At first glance it seems that definition (2), although satisfying requirements (i) and (ii), has a serious flaw. Namely the property of a sequence that it is a Cauchy sequence depends not only on its terms, but also on some other points which need not belong to it. This allows sometimes for a Cauchy sequence, and even for a convergent one, to be rendered a non-Cauchy sequence by removing from the space some point (or points), for instance, the limit point in the case that the sequence converges—which is impossible in the metric case. Nevertheless the following simple example shows that in the case of quasi-metric spaces this situation should be considered as justified.

Example 2. Let (X, d) be the quasi-metric space with $X = \{x_n \mid n = 0, 1, 2, \ldots\}$ and

$$d(x_m, x_n) = \begin{cases} 1 & \text{for } m > 0 \text{ and } m \neq n, \\ 1/n & \text{for } m = 0 \text{ and } n \neq 0, \\ 0 & \text{for } m = n. \end{cases}$$

Suppose we are given a definition of Cauchy sequence satisfying (i) and (ii). Then the sequence $\{x_n \mid n = 1, 2, \ldots\}$ converges to x_0 and therefore, in view of (i), is a Cauchy sequence. However by removing the point x_0 from the space X one obtains a discrete metric space in which the same sequence, according to (ii), must be non-Cauchy.

On the other hand, however, as far as it is known to the author, no convenient construction of a completion based on definition (2) and satisfying (iii) or (iv) has been found up to now. Another, nonformal objection to this definition is illustrated by the following example.

Example 3. (Sorgenfrey line). Let R be the real line equipped with the quasi-metric

$$d_S(x, y) = \begin{cases} y - x & \text{if } x \leqslant y, \\ 1 & \text{if } x > y. \end{cases}$$

The topology of the quasi-metric space (R, d_S) has, for any $x \in X$, the collection $\{[x, x+r) \mid r > 0\}$ for a local base at the point x. This topology, as is known, is nonmetrizable.

Now one sees that, for instance, the sequence $\{-1/n\}$, although nonconvergent, is a Cauchy sequence in the sense of (2). However, in view of the special character of the topology on the space (R, d_S), it seems very inconvenient to regard this sequence as a potentially convergent one, i.e. as one that could be made convergent by completing the space.

In order to eliminate this objection another concept of Cauchy sequence is proposed here which also enables us to realize the program outlined above under a relatively simple additional assumption about the quasi-metric.

1. Cauchy sequences

Here we begin with the following:

Definition 1. A sequence $\{x_n\}$ in the quasi-metric space (X, d) is called *Cauchy sequence* provided that for any natural number k there exist a $y_k \in X$ and an N_k such that $d(y_m, x_n) < 1/k$ whenever $m, n > N_k$.

Further, when two sequences $\{x'_m\}$ and $\{x''_n\}$ are given in a quasi-metric space (X, d), we will write

$$\lim_{m,n} d(x'_m, x''_n) = r$$

if for any $\varepsilon > 0$ there is an N_ε such that $|d(x'_m, x''_n) - r| < \varepsilon$ when $m, n > N_\varepsilon$.

In particular the requirement of Definition 1 could be written as

$$\lim_{m,n} d(y_m, x_n) = 0.$$

(Note that the requirement in definition (1) of Cauchy sequence for metric spaces can be written as $\lim_{m,n} d(x_m, x_n) = 0$.)

In the sequel the term *Cauchy sequence* will be used in the sense of Definition 1.

Any sequence $\{y_m\}$ which satisfies the condition of Definition 1 with reference to a Cauchy sequence $\{x_n\}$, i.e. for which $\lim_{m,n} d(y_m, x_n) = 0$, will be called *cosequence* to $\{x_n\}$.

One observes immediately that requirements (i) and (ii) from the introduction are now fulfilled. This is seen by the following Propositions 1 and 3.

Proposition 1. *Every convergent sequence in a quasi-metric space (X, d) is a Cauchy sequence.*

Proof. If x_n is a convergent sequence in (X, d) and $x_n \to x$, then the condition of Definition 1 is satisfied by letting $y_k = x$ for $k = 1, 2, \ldots$. □

Proposition 2. *Every subsequence of a Cauchy sequence is a Caucy sequence.*

Proof. Obvious. □

Proposition 3. *If (X, d) is a metric space, then Definition* 1 *is equivalent to the usual definition of Cauchy sequence for metric spaces.*

Proof. Let (X, d) be a metric space and $\{x_n\}$ be a Cauchy sequence in the sense of the standard definition. Then for any $\varepsilon > 0$ there is an N_ε such that $d(x_m, x_n) < \varepsilon$ when $m, n > N_\varepsilon$. It is clear that $\{x_n\}$ is a cosequence to itself and so the condition of Definition 1 is fulfilled.

Now let $\{x_n\}$ be a Cauchy sequence in (X, d) in the sense of Definition 1 and let $\{y_m\}$ be a cosequence to $\{x_n\}$. If $\varepsilon > 0$ and $d(y_m, x_n) < \frac{1}{2}\varepsilon$ for $m, n > N_\varepsilon$, then

$$d(x_m, x_n) \leqslant d(x_m, y_m) + d(y_m, x_n) < \varepsilon$$

for $m, n > N_\varepsilon$. □

Now we adopt the following standard.

Definition 2. A quasi-metric space is called *complete* if every Cauchy sequence is convergent.

One verifies that, for instance, the Sorgenfrey line from Example 3 is a complete quasi-metric space. The so called Kofner plane (cf. [1, p. 155]) and Pixley–Roy space (cf. [1, p. 179]), considered as quasi-metric spaces, are also complete. Another example of a complete quasi-metric space is given below.

Example 4. Recall that a real function f defined on an interval $[a, b]$ on the real line is called upper semi-continuous if for every $x \in [a, b]$ and any $\varepsilon > 0$ there is a $\delta > 0$ (depending on x and ε) such that $f(y) < f(x) + \varepsilon$ whenever $|x - y| < \delta$.

Let $C^+([a, b])$ denote the collection of all upper semi-continuous functions defined on $[a, b]$ and define a quasi-metric d_{C^+} on $C^+([a, b])$ as follows

$$d_{C^+}(f, g) = \begin{cases} \min\{\sup\{g(x) - f(x) \mid x \in [a, b]\}, 1\} & \text{if } f \leqslant g, \\ 1 \quad \text{if } f(x) > g(x) & \text{for some } x \in [a, b], \end{cases}$$

where $f \leqslant g$ means that $f(x) \leqslant g(x)$ for each $x \in [a, b]$.

Then one checks that the quasi-metric space $(C^+([a, b]), d_{C^+})$ is complete. □

Note that if one considers the collection $C([a, b])$ of all continuous real functions defined on $[a, b]$ (instead of the upper semi-continuous ones) together with the quasi-metric d_{C^+} from the last example, then the quasi-metric space $(C([a, b]), d_{C^+})$ is also complete.

Remark 1. It is to be noted that a complete subspace of a quasi-metric space may fail to be closed. Indeed the interval $(0, 1)$ in the space (R, d_S) of Example 3 is complete but not closed.

Two quasi-metrics d' and d'' defined on a set X are said to be equivalent if they induce the same quasi-uniformity on X; i.e. if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $d'(x, y) < \delta$ implies $d''(x, y) < \varepsilon$ and $d''(x, y) < \delta$ implies $d'(x, y) < \varepsilon$. For every quasi-metric space (X, d), the quasi-metric d is equivalent to a bounded quasi-metric d'; it suffices to let $d'(x, y) = \min(d(x, y), 1)$.

If $\{(X_i, d_i)\}$ is a finite or countable family of bounded quasi-metric spaces and $d_i(x_i, y_i) \leq b_i$ for $x_i, y_i \in X_i$, then a bounded quasi-metric d in the Cartesian product $\prod_i X_i$ is defined by the equality

$$d(x, y) = \sum_i \frac{1}{2^i b_i} d_i(x_i, y_i) \tag{3}$$

(where $x = \{x_i\}$ and $y = \{y_i\}$ are two points of $\prod_i X_i$). The quasi-uniformity on $\prod_i X_i$ which is induced by d is the product of the quasi-uniformities induced on X_i by d_i. One lets usually $(\prod_i X_i, d) = \prod_i (X_i, d_i)$.

One easily proves:

Theorem 1. *If $\{(X_i, d_i)\}$ is a finite or countable family of complete bounded quasi-metric spaces, then the product $\prod_i (X_i, d_i)$ is complete.*

Proof. Let $\{x^n\}$ be a Cauchy sequence in the product space (X, d), where $X = \prod_i X_i$ and d is defined by (3), and let $x^n = \{x_i^n \mid i = 1, 2, \ldots\}$. If $y^m = \{y_i^m \mid i = 1, 2, \ldots\}$ and $\{y^m\}$ is a cosequence to $\{x^n\}$, then for any $\varepsilon > 0$ there is an N_ε such that $d(y^m, x^n) < \varepsilon$, i.e.

$$\sum_i \frac{1}{2^i b_i} d_i(y_i^m, x_i^n) < \varepsilon$$

when $m, n > N_\varepsilon$. Hence one concludes that, for each fixed i, the sequence $\{x_i^n \mid n = 1, 2, \ldots\}$ is a Cauchy sequence and therefore a convergent one in (X_i, d_i). For each i, let x_i denote the limit of x_i^n in X_i and let $x = \{x_i \mid i = 1, 2, \ldots\}$. Then it is easy to check that $x^n \to x$ in (X, d). $\square$

As usual we use the following:

Definition 3. A *completion* of a quasi-metric space (X, d) is a complete quasi-metric space (X^*, d^*) in which (X, d) can be quasi-isometrically embedded as a dense subspace.

Remark 2. A quasi-metric space could have more than one completion. Moreover, a complete quasi-metric space could be densely embedded in another one. For instance, the interval $(0, 1)$, regarded as a subspace of the space (R, d_S) from Example 3, is complete and is dense in the complete subspace $[0, 1)$ of the same space.

2. Completing

Now we turn to the construction of a completion of a given quasi-metric space. As shown in the Introduction, in order to fulfil requirements (iii) and (iv), we are obliged to restrict ourselves by considering some suitably defined subclass of the class of all T_2-quasi-metric spaces.

In the sequel we consider only those quasi-metric spaces (X, d) whose quasi-metric d satisfies the following supplementary condition:

(B) If $\{x'_n\}$ and $\{x''_m\}$ are two sequences in (X, d) and $x', x'' \in X$, then from

$$d(x', x'_n) \leqslant r' \quad \text{for each } n, \qquad d(x''_m, x'') \leqslant r'' \quad \text{for each } m$$

and

$$\lim_{m,n} d(x''_m, x'_n) = 0$$

it follows that

$$d(x', x'') \leqslant r' + r''.$$

Every quasi-metric d satisfying condition (B) will be called *balanced quasi-metric* or *B-quasi-metric*, the quasi-metric spaces with balanced quasi-metrics will be called *balanced quasi-metric spaces* or *B-quasi-metric spaces* and their class will be denoted by B.

Let us note that every metric d obviously satisfies condition (B). Thus the class of all metric spaces is a subclass of the class B. On the other hand there exist non-metrizable quasi-metric spaces belonging to the class B. Such are the Sorgenfrey line (Example 3) and the space $(C^+([a, b]), d_{C^+})$ from Example 4, as well as the Kofner plane and the Pixley–Roy space, mentioned in the preceding Section.

The balanced quasi-metrics possess a number of useful properties.

Lemma 1. *For a B-quasi-metric d we have:*

$$\text{if } \lim d(x, x_n) = 0 \text{ and } d(y, x_n) \leqslant r \quad \text{for } n = 1, 2, \ldots,$$
$$\text{then } d(y, x) \leqslant r; \tag{4}$$
$$\text{if } \lim d(x_n, x) = 0 \text{ and } d(x_n, y) \leqslant r \quad \text{for } n = 1, 2, \ldots,$$
$$\text{then } d(x, y) \leqslant r. \tag{5}$$

Proof. Assertion (4) is obtained from (B) by letting $x' = y$, $x'_n = x_n$, $x''_m = x'' = x$, and (5)—by letting $x' = x'_n = x$, $x''_m = x_m$, $x'' = y$. □

Lemma 2. *Any B-quasi-metric d has the properties:*

$$\text{if } \lim d(x, x_n) = 0 \text{ and } \lim d(y, x_n) = 0 \quad \text{then } x = y; \tag{6}$$
$$\text{if } \lim d(x_n, x) = 0 \text{ and } \lim d(x_n, y) = 0 \quad \text{then } x = y. \tag{7}$$

Proof. Let $\lim d(x, x_n) = 0$ and $\lim d(y, x_n) = 0$. For any $\varepsilon > 0$ there is an N such that $d(y, x_n) < \varepsilon$ for $n > N$. Then, according to (4), we have $d(y, x) \leqslant \varepsilon$, hence $d(y, x) = 0$, i.e. $x = y$. So (6) holds. Assertion (7) is obtained analogously by means of (5). □

Corollary. *Every B-quasi-metric is a T_2-quasi-metric.*

Lemma 3. *If d is a B-quasi-metric, then*

$$\lim d(x_n, x) = 0, \quad \lim d(y, y_m) = 0 \quad \text{and} \\ d(x_n, y_m) \leq r \quad \text{for } n = 1, 2, \ldots, \quad m = 1, 2, \ldots \tag{8}$$

imply $d(x, y) \leq r$.

Proof. Let us fix m and consider the sequence $\{x_n\}$. From $\lim d(x_n, x) = 0$ and $d(x_n, y_m) \leq r$ for $n = 1, 2, \ldots$ it follows, according to (5), that $d(y, x_m) \leq r$. Then from $d(x, y_m) \leq r$ for $m = 1, 2, \ldots$ and $\lim d(y, y_m) = 0$ one concludes, by (4), that $d(x, y) \leq r$. □

Lemma 4. *For a B-quasi-metric d we have*:

$$\text{if } \lim d(x_n, x) = 0 \text{ and } \lim d(y, y_m) = 0, \\ \text{then } \lim_{n,m} d(x_n, y_m) = d(x, y). \tag{9}$$

Proof. Let ε be an arbitrary positive number. Because of the inequality

$$d(x_n, y_m) \leq d(x_n, x) + d(x, y) + d(y, y_m)$$

there is an N' such that

$$d(x_n, y_m) < d(x, y) + \varepsilon$$

for $n, m > N'$. On the other hand, if one supposes that

$$d(x_{n_k}, y_{m_s}) \leq d(x, y) - \varepsilon$$

for each k and s, where $\{x_{n_k}\}$ and $\{y_{m_s}\}$ are some subsequences of the sequences $\{x_n\}$ and $\{y_m\}$, respectively, then it follows from Lemma 3 that $d(x, y) \leq d(x, y) - \varepsilon$, a contradiction. Therefore there exists an N'' such that

$$d(x_n, y_m) > d(x, y) - \varepsilon$$

for $n, m > N''$. Consequently, we have $\lim_{n,m} d(x_n, y_m) = d(x, y)$. □

By means of (9) one gets immediately the following

Corollary 1. *For any B-quasi-metric d we have*:

$$\text{if } \lim d(x_n, x) = 0 \quad \text{then } \lim d(x_n, y) = d(x, y) \text{ for any } y \in X; \tag{10}$$

$$\text{if } \lim d(x, x_n) = 0 \quad \text{then } \lim d(y, x_n) = d(y, x) \text{ for any } y \in X. \tag{11}$$

Assertion (11) yields:

Corollary 2. *If $(X, d) \in B$, then, for any $x \in X$, the equality*

$$d_x(y) = d(x, y)$$

gives a continuous function d_x, defined on X.

From Corollary 2 there follows:

Corollary 3. *Every B-quasi-metrizable topological space is completely regular.*

Now let us introduce the following definition.

Definition 4. Two Cauchy sequences $\{x'_n\}$ and $\{x''_m\}$ in a B-quasi-metric space (X, d) are called *equivalent* if every cosequence to $\{x'_n\}$ is a cosequence to $\{x''_m\}$ and vice versa.

In the sequel of this section we assume that a quasi-metric space (X, d) belonging to the class B is given. This space, as a rule, will not be mentioned explicitly in the fomulation of the following Definitions, Lemmas and Propositions. (Nevertheless this convention does not concern the formulation of the Theorems.)

Lemma 5. *Let $x_n \to x$. Then $\{y_m\}$ is a cosequence to $\{x_n\}$ if and only if* $\lim d(y_m, x) = 0$.

Proof. If $\lim d(y_m, x) = 0$, then the inequality

$$d(y_m, x_n) \leq d(y_m, x) + d(x, x_n)$$

implies $\lim_{m,n} d(y_m, x_n) = 0$; therefore $\{y_m\}$ is a cosequence to $\{x_n\}$.

Conversely, let $\{y_m\}$ be a cosequence to $\{x_n\}$. For any $\varepsilon > 0$ there is some N such that $d(y_m, x_n) < \varepsilon$ for $m, n > N$. Let us fix an m larger than N. Since $\lim d(x, x_n) = 0$, it follows from (4) that $d(y_m, x) \leq \varepsilon$ for $m > N$. □

Lemma 6. *If $\{x'_n\}$ and $\{x''_m\}$ are two Cauchy sequences with a common cosequence $\{y_k\}$ and $x'_n \to x$, then $x''_m \to x$.*

Proof. Let $\varepsilon > 0$. There is an N such that $d(y_k, x''_m) < \varepsilon$ for $k, m > N$. Fix an m larger than N. Since, according to Lemma 5, we have $\lim d(y_k, x) = 0$, it follows from (5) in Lemma 1 that $d(x, x''_m) \leq \varepsilon$ for $m > N$. □

From Lemma 6 one obtains immediately:

Proposition 4. *If* $\{x'_n\}$ *and* $\{x''_m\}$ *are two equivalent Cauchy sequences and* $x'_n \to x$, *then* $x''_m \to x$.

Then we have the following very useful proposition.

Proposition 5. *If two Cauchy sequences* $\{x'_n\}$ *and* $\{x''_m\}$ *have a common cosequence* $\{y'_p\}$, *then they are equivalent.*

Proof. Let $\{y''_q\}$ be a cosequence to $\{x'_n\}$ and let $\varepsilon > 0$. There exists an N such that $d(y'_p, x''_m) < \frac{1}{2}\varepsilon$ and $d(y''_q, x'_n) < \frac{1}{2}\varepsilon$ for $p, q, m, n > N$. Since $\lim_{p,n} d(y'_p, x'_n) = 0$, it follows from (B) that $d(y''_q, x''_m) \leq \varepsilon$ for $q, m > N$. Consequently $\{y''_q\}$ is a cosequence to $\{x''_m\}$, and so any cosequence to the Cauchy sequence $\{x'_n\}$ is a cosequence to the Cauchy sequence $\{x''_m\}$. The converse follows by the symmetry of the argument. □

This proposition allows us to speak of *cosequences to an equivalence class* of Cauchy sequences (instead of cosequences to a Cauchy sequence).

From Proposition 5 there follows:

Proposition 6. *Every Cauchy sequence is equivalent to any of its subsequences.*

Propositions 4 and 6 yield:

Proposition 7. *The collection of all sequences convergent to a point* $x \in X$ *forms an equivalence class of Cauchy sequences.*

Now let X^* denote the collection of all equivalence classes of Cauchy sequences in the given B-quasi-metric space (X, d).

For every $x \in X$ we let

$$\alpha(x) = \{\{x_n\} \mid x_n \to x\}. \tag{12}$$

Then, according to Proposition 7, $\alpha(x) \in X^*$, and so a mapping

$$\alpha\colon\ X \to X^*$$

is defined.

Further we provide X^* with a quasi-metric d^* by means of the following definition.

Definition 5. Let $\xi', \xi'' \in X$, r be a non-negative real number, $\{y'_m\}$ be a cosequence to the class ξ' and $\{x''_n\} \in \xi''$. We let

$$d^*(\xi', \xi'') \leq r \tag{13}$$

if for any $\varepsilon > 0$ there exists an N_ε such that

$$d(y'_m, x''_n) < r + \varepsilon$$

when $m, n > N_\varepsilon$. Then we let

$$d^*(\xi', \xi'') = \inf\{r \mid d^*(\xi', \xi'') \leq r\}. \tag{14}$$

This definition is justified by the following proposition.

Proposition 8. *The validity of the inequality* (13) *in Definition* 5 *depends only on* ξ', ξ'' *and* r; *it does not depend on the choice of the sequences* $\{y'_m\}$ *and* $\{x''_n\}$.

Proof. Let $\{x''_k\}$ and $\{x^{**}_s\}$ be two Cauchy sequences of the class ξ'' and $\{y'_p\}$ and $\{y^*_q\}$ be two cosequences to the class ξ'. Suppose that for any $\varepsilon > 0$ there exists an N such that

$$d(y'_p, x''_k) < r + \varepsilon$$

for $p, k > N$.

Further, let $\{x'_n\}$ be a Cauchy sequence of the class ξ' and $\{y''_m\}$ be a cosequence to the class ξ''.

Assume that ε^* is an arbitrary positive number and choose an ε' so that $0 < \varepsilon' < \varepsilon^*$. There is an N such that

$$d(y'_p, x''_k) < r + \varepsilon' \tag{15}$$

for $p, k > N'$. Let $0 < \varepsilon < \frac{1}{2}(\varepsilon^* - \varepsilon')$ and let us choose an N_1 in such a manner that

$$d(y^*_q, x'_n) < \varepsilon \tag{16}$$

when $q, n > N_1$. Fix a k larger than N' and a q larger than N_1. Since we have $\lim_{p,n} d(y'_p, x'_n) = 0$, the inequalities (15) and (16) together with the condition (B) yield

$$d(y^*_q, x''_k) \leq r + \varepsilon' + \varepsilon. \tag{17}$$

So the inequality (17) holds for $k > N'$, $q > N_1$.

Analogously, choosing an N_2 so that

$$d(y''_m, x^{**}_s) < \varepsilon \tag{18}$$

for $m, s > N_2$ and having in mind that $\lim_{m,k} d(y''_m, x''_k) = 0$, one concludes by means of (17), (18) and (B) that

$$d(y^*_q, x^{**}_s) \leq r + \varepsilon' + 2\varepsilon < r + \varepsilon^*$$

for $q, s > N^* = \max(N_1, N_2)$.

So if the inequality

$$d^*(\xi', \xi'') \leq r$$

is verified (in the sense of Definition 5) by using the sequences $\{y'_p\}$ and $\{x''_k\}$, it is also verified by using the sequences $\{y^*_q\}$ and $\{x^{**}_s\}$. □

It is to be noted also that inequality (13) and equality (14) in Definition 5 are consistent. The obvious verification is left to the reader.

On the other hand we have the following useful proposition.

Proposition 9. *Let $\xi', \xi'' \in X^*$, $\{y'_m\}$ be a cosequence to the class ξ' and $\{x''_n\}$ be a Cauchy sequence of the class ξ''. Then*

$$d^*(\xi', \xi'') = \lim_{m,n} d(y'_m, x''_n).$$

Proof. Let $d^*(\xi', \xi'') = r$. For any $\varepsilon > 0$ there exists an N' such that

$$d(y'_m, x''_n) < r + \varepsilon$$

for $m, n > N'$. Suppose there exist a subsequence $\{y'_{m_k}\}$ of $\{y'_m\}$ and a subsequence $\{x''_{n_s}\}$ of $\{x''_n\}$ such that $d(y'_{m_k}, x''_{n_s}) \leqslant r - \varepsilon$. Then, since $\{y'_{m_k}\}$ is a cosequence to the class ξ' and $\{x''_{n_s}\} \in \xi''$, it follows, according to Definition 5, that $d^*(\xi', \xi'') \leqslant r - \varepsilon$, a contradiction. Therefore there is some N'' such that

$$d(y'_m, x''_n) > r - \varepsilon$$

for $m, n > N''$. So we obtain $\lim_{m,n} d(y'_m, x''_n) = r$. $\square$

Proposition 10. *d^* is a quasi-metric.*

Proof. (a) Clearly we have $d^*(\xi', \xi'') \geqslant 0$ for $\xi', \xi'' \in X$. From Proposition 9 it follows also immediately that $d^*(\xi, \xi) = 0$ for any $\xi \in X^*$. On the other hand, if $\xi', \xi'' \in X$, $\{x''_n\} \in \xi''$ and $\{y'_m\}$ is a cosequence to the class ξ', and if $d(\xi', \xi'') = 0$, then by Proposition 9 $\{y'_m\}$ is a cosequence to $\{x''_n\}$. Thus ξ' and ξ'' possess a common cosequence, and so by Proposition 5, we have $\xi' = \xi''$.

(b) Let $\xi', \xi'', \xi''' \in X^*$, $d^*(\xi', \xi'') = r_1$, $d(\xi'', \xi''') = r_2$, $\{y'_k\}$ be a cosequence to the class ξ', $\{x''_n\} \in \xi''$, $\{y''_m\}$ be a cosequence to the class ξ'' and $\{x'''_s\} \in \xi'''$. If $\varepsilon > 0$, there are some N_1 and N_2 such that

$$d(y'_k, x''_n) < r_1 + \tfrac{1}{2}\varepsilon \quad \text{for } k, n > N_1$$

and

$$d(y''_m, x'''_s) < r_2 + \tfrac{1}{2}\varepsilon \quad \text{for } m, s > N_2.$$

Since $\lim_{m,n} d(y''_m, x''_n) = 0$, it follows from (B) that

$$d(y'_k, x'''_s) \leqslant r_1 + r_2 + \varepsilon$$

for $k, s > N = \max(N_1, N_2)$. Consequently, according to Definition 5, we have

$$d^*(\xi', \xi''') \leqslant r_1 + r_2 = d^*(\xi', \xi'') + d^*(\xi'', \xi'''). \quad \square$$

Proposition 11. *For $x', x'' \in X$*

$$d^*(\alpha(x'), \alpha(x'')) = d(x', x'').$$

Proof. It suffices to use the sequence $x', x', \ldots$ as a cosequence to the class $\alpha(x')$ and the sequence $x'', x'', \ldots$ as a Cauchy sequence of the class $\alpha(x'')$ and to apply Proposition 9. □

Proposition 11 establishes that the mapping

$$\alpha:\ (X, d) \to (X^*, d^*),$$

defined earlier by (12), is a quasi-metric embedding.

Proposition 12. *For any* $\xi \in X^*$:

(a) $\{x_n\} \in \xi$ *if and only if* $\lim d^*(\xi, \alpha(x_n)) = 0$;

(b) $\{y_m\}$ *is a cosequence to the class* ξ *if and only if* $\lim d(\alpha(y_m), \xi) = 0$.

Proof. (a′) Let $\{x_n\} \in \xi$ and $\{y_m\}$ be a cosequence to the class ξ. For every $\varepsilon > 0$ there is an N such that

$$d(y_m, x_n) < \varepsilon$$

for $m, n > N$. Fix an $n > N$ and consider the sequence $x_n, x_n, \ldots$ as a Cauchy sequence of the class $\alpha(x_n)$. Then Proposition 9 yields

$$d^*(\xi, \alpha(x_n)) \leqslant \varepsilon$$

for $n > N$. Consequently $\lim d^*(\xi, \alpha(x_n)) = 0$.

(b′) By the same assumptions and notation as in (a′) one proves in an analogous manner that

$$d^*(\alpha(y_m), \xi) = 0.$$

(a″) Now let $\lim d^*(\xi, \alpha(x_n)) = 0$ and $\{y_m\}$ be a cosequence to the class ξ. According to (b′) we have $\lim d^*(\alpha(y_m), \xi) = 0$. Then by means of the inequality

$$d^*(\alpha(y_m), \alpha(x_n)) \leqslant d^*(\alpha(y_m), \xi) + d^*(\xi, \alpha(x_n))$$

one obtains

$$\lim_{m,n} d^*(\alpha(y_m), \alpha(x_n)) = 0,$$

In view of Proposition 11, we have $\lim_{m,n} d(y_m, x_n) = 0$. So $\{y_m\}$ is a cosequence to the Cauchy sequence $\{x_n\}$. Consequently $\{x_n\} \in \xi$.

(b″) Finally, let $\lim d^*(\alpha(y_m), \xi) = 0$ and let $\{x_n\} \in \xi$. Then using (a′) and reasoning as in (a″) one proves that $\{y_m\}$ is a cosequence to the Cauchy sequence $\{x_n\}$, i.e. to the class ξ.

The considerations in (a′) and (a″) yield (a), and those in (b′) and (b″) yield (b). □

Proposition 13. *The quasi-metric d^* is a B-quasi-metric in X^*.*

Proof. Let ξ', $\xi'' \in X^*$ and $\{\xi'_n\}$ and $\{\xi''_m\}$ be two sequences in X^* such that

$$d^*(\xi', \xi'_n) \leq r' \quad \text{for } n = 1, 2, \ldots,$$

$$d^*(\xi''_m, \xi'') \leq r'' \quad \text{for } m = 1, 2, \ldots$$

and

$$\lim_{m,n} d^*(\xi''_m, \xi'_n) = 0.$$

Suppose that, for $n = 1, 2, \ldots$, $\{x^n_p \mid p = 1, 2, \ldots\}$ is a Cauchy sequence of the class ξ'_n and, for $m = 1, 2, \ldots$, $\{y^m_q \mid q = 1, 2, \ldots\}$ is a cosequence to the class ξ''_m. For any natural number n there exists an $x^n_{p_n} = x_n$ such that

$$d^*(\xi'_n, \alpha(x_n)) < 1/n.$$

Analogously for any m one finds a $y^m_{q_m} = y_m$ such that

$$d^*(\alpha(y_m), \xi''_m) < 1/m.$$

Then we have

$$\begin{aligned} d(y_m, x_n) &= d^*(\alpha(y_m), \alpha(x_n)) \\ &\leq d^*(\alpha(y_m), \xi''_m) + d^*(\xi''_m, \xi'_n) + d^*(\xi'_n, \alpha(x_n)), \end{aligned}$$

hence

$$\lim_{m,n} d(y_m, x_n) = 0. \tag{19}$$

Fix a cosequence $\{y'_k\}$ to the class ξ' and a Cauchy sequence $\{x''_s\}$ of the class ξ''. We have

$$\begin{aligned} d(y'_k, x_n) &= d^*(\alpha(y'_k), \alpha(x_n)) \\ &\leq d^*(\alpha(y'_k), \xi') + d^*(\xi', \xi'_n) + d^*(\xi'_n, \alpha(x_n)) \\ &\leq d^*(\alpha(y'_k), \xi') + r' + 1/n. \end{aligned}$$

Let $\varepsilon > 0$. Since $\lim d^*(\alpha(y'_k), \xi') = 0$, there is an N_1 such that

$$d(y'_k, x_n) < r' + \tfrac{1}{2}\varepsilon \tag{20}$$

for $k, n > N_1$. Analogously one concludes that there exists an N_2 such that

$$d(y_m, x''_s) < r'' + \tfrac{1}{2}\varepsilon \tag{21}$$

for $m, s > N_2$. Then (19), (20) and (21) together with the condition (B), satisfied by the quasi-metric d, yield

$$d(y'_k, x''_s) \leq r' + r'' + \varepsilon$$

for $k, s > N = \max(N_1, N_2)$. Hence

$$d^*(\xi', \xi'') \leqslant r' + r''.$$

So the quasi-metric d^* satisfies the condition (B) (in which d is replaced by d^*). $\square$

Corollary. *The quasi-metric d^* possesses the properties* (4), (5), (6), (7), (8), (9), (10) *and* (11) (*with d replaced by d^**).

Proposition 14. *The quasi-metric space (X^*, d^*) is complete.*

Proof. Let $\{\xi_n\}$ be a Cauchy sequence in the space (X^*, d^*) and $\{\eta_m\}$ be a cosequence to $\{\xi_n\}$. For each n let $\{x_i^n \mid i = 1, 2, \ldots\}$ be a Cauchy sequence in the space (X, d) belonging to the class ξ_n and, for each m, let $\{y_j^m \mid j = 1, 2, \ldots\}$ be a cosequence to the class η_m. For every natural number n there exists, according to Proposition 12 (assertion (a)), an i_n such that

$$d^*(\xi_n, \alpha(x_i^n)) < 1/n$$

for $i \geqslant i_n$. Analogously for every m there is a j_m such that

$$d^*(\alpha(y_j^m), \eta_m) < 1/m$$

for $j \geqslant j_m$. Letting $x_n = x_{i_n}^n$ and $y_m = y_{j_m}^m$, we have

$$\begin{aligned} d(y_m, x_n) &= d^*(\alpha(y_m), \alpha(x_n)) \\ &\leqslant d^*(\alpha(y_m), \eta_m) + d^*(\eta_m, \xi_n) + d^*(\xi_n, \alpha(x_n)) \\ &< 1/m + d^*(\eta_m, \xi_n) + 1/n. \end{aligned}$$

Hence for every $\varepsilon > 0$ one can find an N such that

$$d(y_m, x_n) < \varepsilon$$

for $m, n > N$.

Therefore $\{x_n\}$ is a Cauchy sequence in the space (X, d) and $\{y_m\}$ is a cosequence to $\{x_n\}$. Let ξ be the equivalence class containing $\{x_n\}$. We will show that $\xi_n \to \xi$ in the space (X^*, d^*). For this purpose we first observe that

$$\begin{aligned} d(y_m, x_i^n) &= d^*(\alpha(y_m), \alpha(x_i^n)) \\ &\leqslant d^*(\alpha(y_m), \eta_m) + d^*(\eta_m, \xi_n) + d^*(\xi_n, \alpha(x_i^n)) \\ &< 1/m + d^*(\eta_m, \xi_n) + 1/n \end{aligned}$$

for $i \geqslant i_n$. Let $\varepsilon > 0$. We choose an N_ε in such a manner that $N_\varepsilon > 3/\varepsilon$ and

$$d^*(\eta_m, \xi_n) < \varepsilon/3$$

for $m, n > N_\varepsilon$. Then, fixing an $n > N_\varepsilon$, we have

$$d(y_m, x_i^n) < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon$$

for $m > N_\varepsilon$, $i \geq i_n$. Therefore, according to Definition 5,

$$d^*(\xi, \xi_n) \leq \varepsilon$$

for $n > N_\varepsilon$. Consequently $\lim d^*(\xi, \xi_n) = 0$, i.e. $\xi_n \to \xi$. So the completeness of the space (X^*, d^*) is verified. □

Propositions 10, 11, 12 (assertion (a)), (13) and (14) yield the following:

Theorem 2. *Every quasi-metric space (X, d) of the class B has a completion (X^*, d^*) constructed in a standard manner. The space (X^*, d^*) also belongs to the class B.*

In the sequel the completion (X^*, d^*) constructed above will be called *standard completion* of the space (X, d). Accordingly the quasi-isometric embedding $\alpha : (X, d) \to (X^*, d^*)$, defined by (12), will be called *standard embedding.*

An obvious statement is given by the following proposition.

Proposition 15. *If the B-quasi-metric space (X, d) is complete, then it coincides (up to a quasi-isometry) with its standard completion.*

From the point of view of the purpose posed in the beginning the following two propositions are important.

Proposition 16. *If (X, d) is a metric space, then its standard quasi-metric completion (X^*, d^*) coincides with the usual metric completion of (X, d).*

Proof. According to Proposition 3, in the metric case Cauchy sequences in the sense of Definition 1 are the usual ones. Moreover, the equivalence relation introduced by Definition 4 is then nothing but the usual equivalence relation between Cauchy sequences in metric spaces. Therefore it suffices to verify that in this case d^* is a metric, i.e. that the condition of symmetry

$$d^*(\xi', \xi'') = d^*(\xi'', \xi') \tag{22}$$

is satisfied. But in a metric space every Cauchy sequence is a cosequence to itself. Therefore, by Proposition 9, we have

$$d^*(\xi', \xi'') = \lim_{m,n} d(x'_m, x''_n) \quad \text{and} \quad d^*(\xi'', \xi') = \lim_{n,m} d(x''_n, x'_m)$$

when $\xi', \xi'' \in X^*$, $\{x'_m\} \in \xi'$, $\{x''_n\} \in \xi''$, and the equality (22) follows from the symmetry of the metric d. □

So requirement (iv) from the Introduction is fulfilled.

Proposition 17. *Let (X, d_X) and (Y, d_Y) be two B-quasi-metric spaces. If (Y, d_Y) is complete and $(X, d_X) \subset (Y, d_Y)$, then $(X^*, d_X^*) \subset (Y, d_Y)$. (Here (X^*, d_X^*) is the standard completion of (X, d_X) and the inclusions are understood as quasi-metric embeddings, the second one being an extension of the former.) In particular, for any B-quasi-metric completion (X', d') of (X, d_X) we have $(X^*, d_X^*) \subset (X', d')$.*

Proof. Consider, for simplicity, X as a subset of Y (so that d_X is the restriction of d_Y on $X \times X$). Let $\xi \in X^*$ and $\{x_n\}$ be a Cauchy sequence in (X, d_X) belonging to the class ξ. Then it is also a Cauchy sequence in (Y, d_Y) and therefore converges to a point $y \in Y$, i.e. $\lim d_Y(y, x_n) = 0$. Thereby y is uniquely determined. In fact, if $\{x'_m\}$ is another Cauchy sequence of the class ξ, then $\{x_n\}$ and $\{x'_m\}$ are equivalent in the space (Y, d_Y) and consequently, according to Proposition 4, we have also $\lim d_Y(y, x'_m) = 0$. So a mapping

$$\lambda: X^* \to Y$$

is defined by the condition

$$\lim d_Y(\lambda(\xi), x_n) = 0,$$

where $\xi \in X^*$, $\{x_n\} \in \xi$. In particular, $\lambda(x) = x$ for $x \in X$.

Now let $\xi', \xi'' \in X^*$, $\{x'_p\} \in \xi'$, $\{x''_q\} \in \xi''$ and $\{y'_k\}$ be a cosequence to the class ξ'. Then by Proposition 9 we have

$$\lim_{k,q} d_X(y'_k, x''_q) = d^*_X(\xi', \xi'').$$

On the other hand from $\lim d_Y(\lambda(\xi'), x'_p) = 0$ it follows by Lemma 5 that

$$\lim d_Y(y'_k, \lambda(\xi')) = 0. \tag{23}$$

But

$$\lim d_Y(\lambda(\xi''), x''_q) = 0 \tag{24}$$

and then, according to Lemma 4, (23) and (24) yield

$$\lim_{k,q} d_X(y'_k, x''_q) = \lim_{k,q} d_Y(y'_k, x''_q) = d_Y(\lambda(\xi'), \lambda(\xi'')).$$

Therefore,

$$d^*_X(\xi', \xi'') = d_Y(\lambda(\xi'), \lambda(\xi'')),$$

i.e. the mapping $\lambda : (X^*, d^*_X) \to (Y, d_Y)$ is a quasi-metric embedding. □

Corollary. *If (X, d_X) and (Y, d_Y) are B-quasi-metric spaces and $(X, d_X) \subset (Y, d_Y)$, then for their standard quasi-metric completions we have $(X^*, d^*_X) \subset (Y^*, d^*_Y)$ (both inclusions understood as quasi-metric embeddings).*

Thus the requirement (ii) from the Introduction is also fulfilled and thereby the program sketched in the beginning is carried out.

By means of Theorem 2 one establishes, in addition, the following Theorem.

Theorem 3. *If $\{(X_i, d_i)\}$ is a finite or countable family of bounded B-quasi-metric spaces, then the product $(X, d) = \prod_i (X_i, d_i)$ also belongs to the class B.*

Proof. As mentioned in Section 1, the quasi-metric d in $X = \prod_i X_i$ is defined by (3)

$$d(x, y) = \sum_i \frac{1}{2^i b_i} d_i(x_i, y_i)$$

where $x, y \in X$, $x = \{x_i\}$, $y = \{y_i\}$ and $d_i(x_i', x_i'') \leq b_i$ for $x_i', x_i'' \in X_i$.

We have to show that d satisfies the condition (B). Let

$$d(x', x^n) \leq r', \quad n = 1, 2, \ldots, \qquad d(y^m, x'') \leq r'', \quad m = 1, 2, \ldots$$

and

$$\lim_{m,n} d(y^m, x^n) = 0, \tag{25}$$

where $x', x'', x^n, y^m \in X$, $x' = \{x_i'\}$, $x'' = \{x_i''\}$, $x^n = \{x_i^n \mid i = 1, 2, \ldots\}$, $y^m = \{y_i^m \mid i = 1, 2, \ldots\}$.

Further, for each i, let (X_i^*, d_i^*) be the standard completion of (X_i, d_i). From (3) and (25) it follows that

$$\lim_{m,n} d(y_i^m, x_i^n) = 0, \quad i = 1, 2, \ldots,$$

i.e. that, for each i, $\{x_i^n \mid n = 1, 2, \ldots\}$ is a Cauchy sequence in (X_i, d_i) having $\{y_i^m \mid m = 1, 2, \ldots\}$ as a cosequence and defining a point ξ_i in (X_i^*, d_i^*). Then, if $\alpha_i : X_i \to X_i^*$ is the standard embedding, we have by Proposition 12

$$\lim_m d_i^*(\alpha_i(y_i^m), \xi_i) = 0 \quad \text{and} \quad \lim_n d_i^*(\xi_i, \alpha_i(x_i^n)) = 0$$

for $i = 1, 2, \ldots$ and therefore, according to (10) and (11),

$$\lim_m d_i(y_i^m, x_i'') = \lim_m d_i^*(\alpha_i(y_i^m), \alpha_i(x_i'')) = d_i^*(\xi_i, \alpha_i(x_i''))$$

and

$$\lim_n d_i(x_i', x_i^n) = \lim_n d_i^*(\alpha_i(x_i'), \alpha_i(x_i^n)) = d_i^*(\alpha_i(x_i'), \xi_i).$$

Therefore from

$$d(x', x^n) = \sum_i \frac{1}{2^i b_i} d_i(x_i', x_i^n) \leq r'$$

and

$$d(y^m, x'') = \sum_i \frac{1}{2^i b_i} d_i(y_i^m, x_i'') \leq r''$$

one obtains

$$\sum_i \frac{1}{2^i b_i} d_i^*(\alpha_i(x_i'), \xi_i) \leq r', \qquad \sum_i \frac{1}{2^i b_i} d_i^*(\xi_i, \alpha_i(x_i'')) \leq r''.$$

Hence

$$d(x', x'') = \sum_i \frac{1}{2^i b_i} d_i(x_i', x_i'') = \sum_i \frac{1}{2^i b_i} d_i^*(\alpha_i(x_i'), \alpha_i(x_i''))$$

$$\leq \sum_i \frac{1}{2^i b_i} [d_i^*(\alpha_i(x_i'), \xi_i) + d_i^*(\xi_i, \alpha_i(x_i''))] \leq r' + r''.$$

Thus the quasi-metric d satisfies the condition (B). □

The proof of the following statement is left to the reader.

Proposition 18. *If $\{(X_i, d_i)\}$ is a finite or countable family of bounded B-quasi-metric spaces and, for each i, (X_i^*, d_i^*) is the corresponding standard completion, then the product $\prod_i (X_i^*, d_i^*)$ is quasi-isometric to the standard completion of the product $\prod_i (X_i, d_i)$.*

3. Extension of quasi-uniformly continuous mappings

A mapping

$$f\colon\ (X, d_X) \to (Y, d_Y),$$

where (X, d_X) and (Y, d_Y) are quasi-metric spaces, is called *quasi-uniformly continuous* if for any $\varepsilon > 0$ there is such a $\delta > 0$ that $d_Y(f(x'), f(x'')) < \varepsilon$ whenever $d_X(x', x'') < \delta$.

Lemma 7. *Let (X, d_X) and (Y, d_Y) be (arbitrary) quasi-metric spaces and $f\colon (X, d_X) \to (Y, d_Y)$ be a quasi-uniformly continuous mapping. Then the image $\{f(x_n)\}$ of every Cauchy sequence $\{x_n\}$ in (X, d_X) is a Cauchy sequence in (Y, d_Y). If $\{x_m'\}$ is a cosequence to $\{x_n\}$, then $\{f(x_m')\}$ is a cosequence to $\{f(x_n)\}$.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in the space (X, d_X) and $\{x_m'\}$ be a cosequence to $\{x_n\}$. For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(x', x'') < \delta$ implies $d_Y(f(x'), f(x'')) < \varepsilon$. On the other hand there is an N such that that $d_X(x_m', x_n) < \delta$ when $m, n > N$. Hence $d_Y(f(x_m'), f(x_n)) < \varepsilon$ for $m, n > N$, and so $\{f(x_n)\}$ is a Cauchy sequence in the space (Y, d_Y) with $\{f(x_m')\}$ as a cosequence. □

Theorem 4. *Let (X, d_X) and (Y, d_Y) be two B-quasi-metric spaces and (X^*, d_X^*) and (Y^*, d_Y^*) be their standard completions. Then any quasi-uniformly continuous mapping $f\colon (X, d_X) \to (Y, d_Y)$ has a (unique) quasi-uniformly continuous extension $f^*\colon (X^*, d_X^*) \to (Y^*, d_Y^*)$ (in the sense that $f^* \circ \alpha_X = \alpha_Y \circ f$, where $\alpha_X\colon X \to X^*$ and $\alpha_Y\colon Y \to Y^*$ are the standard embeddings).*

Proof. Let $\xi \in X^*$. According to the preceding Lemma, the equivalence class ξ of Cauchy sequences in the space (X, d_X) defines in a unique manner an equivalence class η of Cauchy sequences in the space (Y, d_Y). Namely, if the class ξ contains the sequence $\{x_n\}$, then the image of each Cauchy sequence of the class ξ belongs to the equivalence class η containing the sequence $\{f(x_n)\}$. Therefore a mapping

$$f^*\colon\ (X^*, d_X^*) \to (Y^*, d_Y^*)$$

is defined by the equality $f^*(\xi) = \eta$, valid when $\{x_n\} \in \xi$ implies $\{f(x_n)\} \in \eta$.

It is obvious that $f^*(\alpha_X(x)) = \alpha_Y(f(x))$ for any $x \in X$. It remains to verify that the mapping f^* is quasi-uniformly continuous. For any $\varepsilon > 0$ there is such a $\delta > 0$ that $d_X(x', x'') < \delta$ implies $d_Y(f(x'), f(x'')) < \varepsilon$. Let $\xi', \xi'' \in X$, $\{x'_m\}$ be a cosequence to the equivalence class ξ' and $\{x''_n\} \in \xi''$. If

$$d_X^*(\xi', \xi'') < \delta \tag{26}$$

then, according to Definition 5, there exists an N such that $d_X(x'_m, x''_n) < \delta$ for $m, n > N$ and then $d_Y(f(x'_m), f(x''_n)) < \varepsilon$. Since $\{f(x'_m)\}$ is a cosequence to the equivalence class $f(\xi')$ in (Y^*, d_Y^*) and $\{f(x''_n)\} \in f(\xi'')$, we have

$$d_Y^*(f(\xi'), f(\xi'')) < \varepsilon. \tag{27}$$

Thus (26) implies (27). $\square$

Theorem 4 and Proposition 15 yield the following.

Corollary. *Let (X, d_X) and (Y, d_Y) be B-quasi-metric spaces and (Y, d_Y) be complete. Then any quasi-uniformly continuous mapping $f\colon (X, d_X) \to (Y, d_Y)$ has a quasi-uniformly continuous extension $f^*\colon (X^*, d_X^*) \to (Y, d_Y)$ over the standard completion (X^*, d_X^*) of (X, d_X) (in the sense that $f^* \circ \alpha_X = f$, where $\alpha_X\colon X \to X^*$ is the standard embedding).*

4. Conjugate completion

As is known, to any quasi-metric d, introduced in a set X, a *conjugate* quasi-metric d^- is related which is defined by the equality

$$d^-(x, y) = d(y, x).$$

The following two Lemmas are obvious.

Lemma 8. *If $(X, d) \in B$, then $(X, d^-) \in B$.*

Lemma 9. *Let $(X, d) \in B$. Then $\{x_n\}$ is a Cauchy sequence in the space (X, d) with $\{y_m\}$ as a cosequence if and only if $\{y_m\}$ is a Cauchy sequence in the space (X, d^-) with $\{x_n\}$ as a cosequence.*

Let us agree, when $\{x_n\}$ is a Cauchy sequence in a quasi-metric space (X, d) and $\{y_m\}$ is a cosequence to $\{x_n\}$, to call the pair $\{\{y_m\}, \{x_n\}\}$ *Cauchy system* in (X, d) and to say that two Cauchy systems $\{\{y'_m\}, \{x'_n\}\}$ and $\{\{y''_p\}, \{x''_q\}\}$ are *equivalent* provided $\{x'_n\}$ and $\{x''_q\}$ are equivalent Cauchy sequences.

In view of Proposition 5 it is clear that two Cauchy systems $\{\{y'_m\}, \{x'_n\}\}$ and $\{\{y''_p\}, \{x''_q\}\}$ in a space $(X, d) \in B$ are equivalent if and only if the pair $\{\{y'_m\}, \{x''_q\}\}$ is a Cauchy system in (X, d), and that in this case $\{\{y''_p\}, \{x'_n\}\}$ is a Cauchy system as well. Therefore, by Lemma 9, there is a one-to-one correspondence between the collection of all equivalence classes of Cauchy sequences in (X, d) and the collection of all equivalence classes of Cauchy sequences in (X, d^-). Moreover, each of them can be identified with the collection of all equivalence classes of Cauchy systems in (X, d) or—what is essentially the same—in (X, d^-). The correspondence among these four collections permits us to denote each of them by X^*. Then one easily verifies the validity of the following

Theorem 5. *Let* $(X, d) \in B$. *Then*:

(a) (X, d) *is complete if and only if* (X, d^-) *is complete*;

(b) *for the standard completions* (X^*, d^*) *and* $(X^*, (d^-)^*)$ *of the conjugate spaces* (X, d) *and* (X, d^-) *we have*

$$(X^*, (d^-)^*) = (X^*, (d^*)^-).$$

References

[1] P. Fletcher and W.F. Lindgren, Quasi-Uniform Spaces (Marcel Dekker, New York and Basel, 1982).

[2] M.G. Murdeshwar and S.A. Naimpally, Quasi-Uniform Topological Spaces (Noordhoff, Groningen, 1966).

[3] I.L. Reilly, P.V. Subrahmanyam and M.K. Vamanamurthy, Cauchy sequences in quasi-pseudo-metric spaces, Monatsh. Math. 93 (1982) 127–140.

[4] J.L. Sieber and W.J. Pervin, Completeness in quasi-uniform spaces, Math. Ann. 158 (1965) 79–81.